# Mid-term test for Quantum Physics 1 - 2013-2014

**Thursday 26 September 2013, 9:00 - 10:00**

<u>READ THIS FIRST</u>:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this test has 3 questions, it continues on the backside of the paper!
- Start each question (number T1, T2,...) on a new side of an answer sheet.
- The test is open book within limits. You are allowed to use the book by Griffiths, the copies from the Feynman book, and one A4 sheet with personal notes, but nothing more than this.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first. The test is only 1 hour.
- When you turn in your problems, please **put your answer sheets in the order T1, T2, T3**...and **staple** them together.

**Useful formulas and constants:**

| | | |
|---|---|---|
| Electron mass | $m_e$ | $= 9.1 \cdot 10^{-31}$ kg |
| Electron charge | $-e$ | $= -1.6 \cdot 10^{-19}$ C |
| Planck's constant | $h$ | $= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs |
| Planck's reduced constant | $\hbar$ | $= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs |

**Problem T1** - ***For this problem, you must write up your answers in Dirac notation.***

Consider a quantum system that contains a charged particle with mass $m$, that has a time-independent Hamiltonian

$$\hat{H} = \hat{T} + \hat{V},$$

where $T$ a kinetic-energy term and $V$ a potential-energy term. With respect to a lowest point in the potential, defined as $V = 0$ J, the two energy eigenstates of this system with the lowest energy are defined by

$$\begin{aligned} \hat{H}|\varphi_1\rangle &= E_1|\varphi_1\rangle \\ \hat{H}|\varphi_2\rangle &= E_2|\varphi_2\rangle \end{aligned} ,$$

where $E_1 < E_2$ the two energy eigenvalues, and $|\varphi_1\rangle$ and $|\varphi_2\rangle$ two orthogonal, normalized energy eigenvectors. The observable $\hat{A}$ is associated with the magnetic dipole moment $A$ of this quantum system. For this system,

$$\langle\varphi_1|\hat{A}|\varphi_1\rangle = 0 \quad , \quad \langle\varphi_2|\hat{A}|\varphi_2\rangle = 0 \quad , \quad \langle\varphi_1|\hat{A}|\varphi_2\rangle = \langle\varphi_2|\hat{A}|\varphi_1\rangle = A_0 \; .$$

Note that the states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are energy eigenvectors, and that they are ***not*** eigenvectors of $\hat{A}$.

**a)** [1 point]

What can you say about the possible values of $E_1$? Discuss the sign, and whether it can be zero. Explain your answer.

**b)** [1 point]

At some time defined as $t = 0$, the state of the system is (with all $c_n$ a complex-valued constant)

$$|\Psi_0\rangle = c_1\,|\varphi_1\rangle + c_2\,|\varphi_2\rangle = \sqrt{\tfrac{2}{5}}\,|\varphi_1\rangle + e^{i\varphi}\sqrt{\tfrac{3}{5}}\,|\varphi_2\rangle \; .$$

Here $\varphi$ (a real number) is the phase of the superposition at $t = 0$. Prove that this is a normalized state.

**Z.O.Z. for further questions**

**c)** [1 points]
Assume that the system is in the state $|\Psi_0\rangle$ (of question **b)**) at $t = 0$. Show that as a function of time $t > 0$, the expectation value for $\langle \hat{A} \rangle$ has oscillations at one frequency only. Determine this frequency and the oscillation amplitude (expressed in the constants that are mentioned above).

**Problem T2**
The position $x$ of a particle is at some time $t = 0$ described by the normalized, real-valued wavefunction

$$\Psi(x) = Ce^{-|x/a|} \quad ,$$

with $a = 3$ nm.

**a)** [1 point]
Make a sketch of both $\Psi(x)$ and the probability density $W(x)$ for the particle's position.

**b)** [1 point]
Show that the state is normalized for $C = 0.577\ \mathrm{nm}^{-1/2}$, and explain the unit of $C$.

**c)** [1 point]
With the particle's wavefunction as sketched, you plan to measure the position $x$. What is the probability for detecting a value in the range $0\ \mathrm{nm} < x < 1\ \mathrm{nm}$?

**d)** [1 point]
You measure the position $x$, with a measurement apparatus that has a resolution of 0.1 nm. You detect the particle at the position $x = 2$ nm. Make a sketch of the probability density $W(x)$ for the particle's position immediately after the measurement. Explain your answer, and the width, height and area of $W(x)$ in your sketch.

**Problem T3** [2 points]
A wide parallel beam of neutrons (only motion in $y$-direction) with a velocity of $v_y = 200$ m/s is incident on a screen with a single narrow slit of width $d$. The direction orthogonal to the beam, along which the slit has a width $d$, is the $x$-direction. Behind this first screen there is a second screen for detection. The distance between the screen with the slit and the detection screen is $l = 5$ m. Using the uncertainty principle, make an estimate for the width of the slit $d$ for which the width $W$ of the image on the detection screen is the narrowest.
**Hint:** For neutrons that just passed the screen, you can assume that for transverse motion in the beam (in $x$-direction), the state of the neutrons is close to a state with minimum uncertainty. The beam will get wider after the screen because the neutrons now have non-zero velocities in $x$-direction. For getting your answer make an estimate for the magnitude of these velocities in $x$-direction.

# Answers to mid-term exam Quantum Physics I
26 Sept 2013

**T1** a) The system has for its lowest energy eigenstates a set of (at least) 2 discrete states ⇒ these correspond to Localized states (confined), where the particle is trapped near the minimum in potential energy V (note, these states are not a free-particle state). $\langle \hat{V} \rangle$ and $\langle \hat{T} \rangle$ must both be larger than 0 J. The particle will have some point energy in the state $|\varphi_1\rangle$, so $E_1 > 0\,J$

b) We need to show that $\langle \psi_s | \psi_s \rangle = 1$, and can use that

$$\begin{cases} \langle \varphi_n | \varphi_m \rangle = 1 & \text{for } n = m \\ \langle \varphi_n | \varphi_m \rangle = 0 & \text{for } n \neq m \end{cases}$$

$$\langle \psi_s | \psi_s \rangle = \left( c_1^* \langle \varphi_1 | + c_2^* \langle \varphi_2 | \right) \left( c_1 |\varphi_1\rangle + c_2 |\varphi_2\rangle \right)$$

$$= |c_1|^2 \langle \varphi_1 | \varphi_1 \rangle + |c_2|^2 \langle \varphi_2 | \varphi_2 \rangle =$$

$$\left|\sqrt{\tfrac{2}{5}}\right|^2 + \left|\sqrt{\tfrac{3}{5}}\right|^2 = \frac{2}{5} + \frac{3}{5} = 1$$

c) $|\psi_0\rangle = c_1 |\varphi_1\rangle + c_2 |\varphi_2\rangle$

Use $\frac{E_i}{\hbar} = \omega_i$

$$|\psi(t)\rangle = \hat{U} |\psi_0\rangle = e^{-\frac{i}{\hbar}\hat{H}t} |\psi_0\rangle$$

$$= e^{-\frac{i}{\hbar}E_1 t} c_1 |\varphi_1\rangle + e^{-\frac{i}{\hbar}E_2 t} c_2 |\varphi_2\rangle$$

$$= e^{-i\omega_1 t} c_1 |\varphi_1\rangle + e^{-i\omega_2 t} c_2 |\varphi_2\rangle .$$

$$\langle \hat{A}(t) \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle$$



$$= \left( e^{+i\omega_1 t} c_1^* \langle \varphi_1 | + e^{+i\omega_2 t} c_2^* \langle \varphi_2 | \right) \hat{A} \left( e^{-i\omega_1 t} c_1 |\varphi_1\rangle + e^{-i\omega_2 t} c_2 |\varphi_2\rangle \right)$$

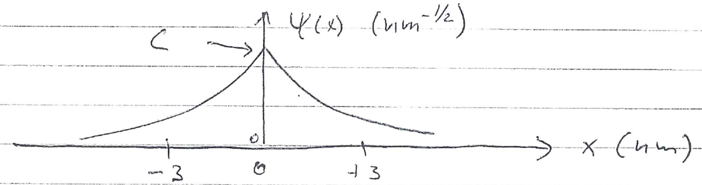
$$= c_1^* c_1 \langle \varphi_1 | \hat{A} | \varphi_1 \rangle + c_2^* c_2 \langle \varphi_2 | \hat{A} | \varphi_2 \rangle$$
$$+ e^{-i(\omega_2 - \omega_1)t} c_1^* c_2 \langle \varphi_1 | \hat{A} | \varphi_2 \rangle + e^{+i(\omega_2 - \omega_1)t} c_2^* c_1 \langle \varphi_2 | A | \varphi_1 \rangle$$

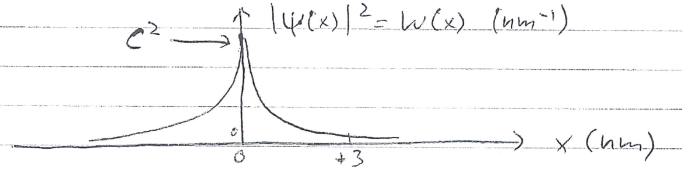
$$= e^{-i(\omega_2 - \omega_1)t} e^{+i\varphi} \sqrt{\frac{6}{25}} A_0 + e^{+i(\omega_2 - \omega_1)t} e^{-i\varphi} \sqrt{\frac{6}{25}} A_0$$

$$= 2\cos\left((\omega_2 - \omega_1)t - \varphi\right) \sqrt{\frac{6}{25}} A_0 = \sqrt{\frac{24}{25}} \cos\left((\omega_2 - \omega_1)t - \varphi\right) A_0$$

So, the amplitude is $\sqrt{\frac{24}{25}} A_0$, and there is only an oscillation at frequency $f = \frac{\omega_2 - \omega_1}{2\pi}$

**T2** a)




b) Normalized if $\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = C^2 \int_{-\infty}^{\infty} e^{-2\left|\frac{x}{a}\right|} dx = 2C^2 \int_0^{\infty} e^{-2\frac{x}{a}} dx = \frac{2C^2 a}{-2} \left[ e^{-2\frac{x}{a}} \right]_0^{\infty}$$

$$= -C^2 a [0 - 1] = C^2 a = 1 \Rightarrow C = \frac{1}{\sqrt{a}} = \frac{1}{\sqrt{3}} \, nm^{-1/2} = 0.58 \, nm^{-1/2}$$

$|\psi(x)|^2 = W(x)$ is a probability density, so $W(x)dx$ is a dimensionless number, so $C$ must be of dimension $\frac{1}{\sqrt{\text{length}}}$.

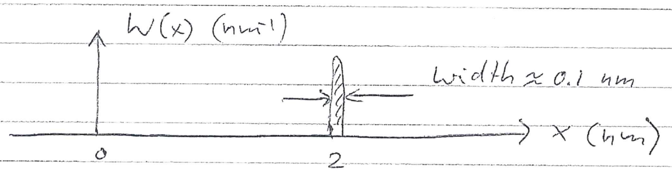
(3)

c) $P(0\,\text{nm} < x < 1\,\text{nm}) = P_{01} = \int_0^{1\,\text{nm}} W(x)\,dx$

$= \int_0^{1\,\text{nm}} C^2 e^{-2\frac{x}{a}}\,dx = \frac{C^2 a}{-2}\left[e^{-2\frac{x}{a}}\right]_0^{1\,\text{nm}}$

$= -\frac{1}{2}\left[e^{-2\frac{1}{3}} - 1\right] = \frac{1}{2}\left(1 - e^{-2/3}\right) \approx 0{,}24$

d)



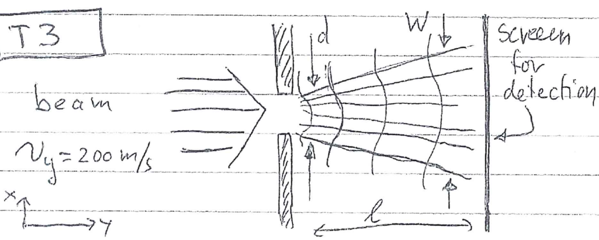

$\int_{-\infty}^{\infty} W(x)\,dx = 1 \Rightarrow$ area $= 1 \Rightarrow$ height is $\frac{1}{0{,}1}$ nm

(Answers using ⎍ or ⋀ are also okay)

T3

beam

$v_y = 200$ m/s




Right after the screen the slit gives

$2\Delta x \approx d \Rightarrow$

$\Delta p_x \approx \frac{\hbar}{4\Delta x} = \frac{\hbar}{4d}$

While flying to the detection screen, the beam will get wider from $d$ to a width $W = d + 2\Delta W$

$\Delta W = \Delta v_x \cdot t = \frac{\Delta p_x}{m} \cdot t$, where $t$ the time of flight between the two screens. $\Rightarrow t = \frac{l}{v_y} \Rightarrow$

$W = d + 2\Delta W = d + 2\frac{\hbar}{4md} \cdot \frac{l}{v_y} \Rightarrow W$ has a minimum for a certain $d$. To find this solve $dW/dd = 0 \Rightarrow$

$1 - \frac{1}{d^2}\frac{\hbar l}{2mv_y} = 0 \Rightarrow d \approx \sqrt{\frac{\hbar l}{2mv_y}}$